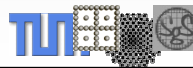


Continuous Models 1: ODE

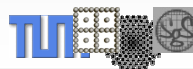
- in scientific computing: *numerical* simulations and, hence, typically *continuous* models
- two big classes:
 - problems *with* a treatment of space (involving *partial* differential equations (PDE))
 - problems *without* a treatment of space (involving *ordinary* differential equations (ODE))
- standard example for the latter: population dynamics
 - development (growth) of populations,
 - either isolated (without external influences)
 - or in coexistence (peaceful or hostile) of different species
 - modelling has a long tradition
 - classical representative: model of Maltus (1798)



Model of Maltus

- one species considered:
 - constant birth rate γ per time unit and individual
 - constant death rate δ per time unit and individual
 - thus, constant growth rate $\lambda = \gamma - \delta$
- development of $p(t)$, the number of individuals:
$$p(t + \Delta t) = p(t) + \lambda p(t) \Delta t$$

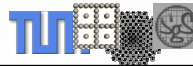
(growth is proportional to size of population and time)
- this leads to the ODE $\dot{p}(t) = \lambda p(t)$ with solution
$$p(t) = p_0 e^{\lambda t} \quad \text{if} \quad p(0) = p_0$$
- note:
 - exponential growth or decrease
 - *discrete* reality, but *continuous* model!



Model Refinement 1

- Is exponential growth realistic?
 - population of the earth between 1700 and 1960: yes!
 - growth rate of about 0.02
 - population doubles in 34.67 years
 - generally: no!
 - limited capacity of the earth, limited resources
 - increasing competition for food, water, or air slows down growth
- refinement following Verhulst and others (19. cent.):
 - population tends towards some saturation limit
 - *linear* birth and death rates (now per time unit only):

$$\gamma(t) = \gamma_0 - \gamma_1 p(t) \quad \delta(t) = \delta_0 + \delta_1 p(t) \quad \gamma_0 > \delta_0 > 0, \gamma_1, \delta_1 > 0$$
 big population decreases birth rate and increases death rate
 - Limit exists if time tends towards infinity!

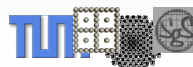


Model Refinement 2

- Is Verhulst's approach realistic?
 - Second derivative of $p(t)$ does not change its sign!
 - S-shape is widespread, however (US-population 1790-1950)
- further refinement in order to obtain S-shape:

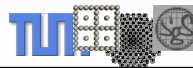
$$\dot{p}(t) = a \cdot p(t) - b \cdot p^2(t), \quad a > b > 0$$
 with solution

$$p(t) = \frac{a \cdot p_0}{b \cdot p_0 + (a - b \cdot p_0)e^{-at}}$$
- discussion:
 - limit for t towards infinity is a/b
 - S-shape for $p_0 < a/b$
 - example USA 1790-1950: $a=0.03134$, $b=0.000000000156$
 - better than our starting point, but still no external influences



Model Refinement 3

- Typically, a is much bigger than b :
 - The quadratic term gets influence only for really big $p(t)$.
 - Why quadratic and not cubic? – That's modelling!
 - justification: an individual is disturbed proportional to $p(t)$
- S-shape growth is called logistic
- For that, other ODEs can be used, too (widespread in modelling of tumours' growth, e.g.):
 - $\dot{p}(t) = \lambda(t) \cdot p(t)$ with some continuous, positive, and decreasing function λ (empirically, growth rates have to decrease)
 - or, even more general, $\dot{p}(t) = f(p(t)) \cdot p(t)$ with some suitable non-negative, decreasing, and vanishing (for increasing t) f



More than one Species

- next step of refinement: consider two species p and q

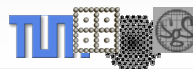
$$\dot{p}(t) = f(p(t), q(t)) \cdot p(t),$$

$$\dot{q}(t) = g(p(t), q(t)) \cdot q(t)$$

f and g defined for positive p and q
- A solution is a limit or stationary situation (no growth)

$$f(\bar{p}, \bar{q}) = g(\bar{p}, \bar{q}) = 0, \quad \bar{p}, \bar{q} > 0$$

and is called *equilibrium* (stable state of the system).
- Is there an equilibrium? If yes, is it *attractive*?
 - theory of ODEs: a sufficient condition are negative real parts of the eigenvalues of the Jacobian of $F(p, q) = (f(p, q)p, g(p, q)q)$
 - We study two special cases: *predator-prey* and *competition* characteristics.



Competition Characteristics

- The species p and q do not „eat“ each other, but both struggle for the same resources:

$$f_p(p, q), f_q(p, q), g_p(p, q), g_q(p, q) < 0 \text{ for } p, q > 0$$

- Sufficient condition for attractive equilibrium reads

$$f_p(\bar{p}, \bar{q}) \cdot g_q(\bar{p}, \bar{q}) - f_q(\bar{p}, \bar{q}) \cdot g_p(\bar{p}, \bar{q}) > 0$$

(p 's influence on p is bigger than on q and vice versa)

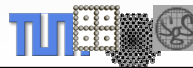
- simple concrete choice for f and g :

$$f(p, q) = a_1 + a_2 \cdot p + a_3 \cdot q, \quad g(p, q) = a_4 + a_5 \cdot p + a_6 \cdot q$$

with (due to our model assumptions)

$$a_1, a_4 > 0, \quad a_2, a_3, a_5, a_6 < 0, \quad a_2 \cdot a_6 > a_3 \cdot a_5$$

- Attractive equil. $f(\bar{p}, \bar{q}) = g(\bar{p}, \bar{q}) = 0, \quad \bar{p}, \bar{q} > 0$ exists!



Predator-Prey Characteristics

- q is „food“ of p , which leads to a different growth behaviour:

$$f_p(p, q), g_p(p, q), g_q(p, q) < 0, \quad f_q(p, q) > 0 \text{ for } p, q > 0$$

$$a_2, a_5, a_6 < 0, \quad a_3 > 0$$

(p , of course, enjoys an increasing population q)

- Sufficient condition for attractive equilibrium is always fulfilled (i.e., if there is an equilibrium, it is attractive).
- classical representative: model of Volterra and Lotka:

$$a_2 = a_6 = 0$$

- no influence of p on p or q on q
- There is an attractive equilibrium (though our sufficient condition is not valid)!

